Math 3236 Statistical Theory

4/20/23

$$\Lambda(\underline{x}) = \frac{\sup_{\theta \in \Omega_0} f_n(\underline{x} \mid \theta)}{\sup_{\theta \in \Omega} f_n(\underline{x} \mid \theta)}$$ likelihood ratio.

$\Omega_0 = \{\theta_0\}$ $\quad$ $\Omega_1 = \{\theta_1\}$

$$f(\underline{x} \mid \theta_0) \Big/ f(\underline{x} \mid \theta_1)$$

$$\Lambda(x) = \frac{f(\underline{x} \mid \theta_0)}{\max\{f(\underline{x} \mid \theta_0), f(\underline{x} \mid \theta_1)\}}$$

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Test for Normal distribution with both $\mu$ and $\sigma^2$ unknown

$$\hat{\sigma}' = \sqrt{\frac{\sum_{i=1}^{N} (X_i - \bar{X})^2}{N-1}}$$

$$\bar{X} = \frac{1}{N} \sum_i X_i$$

$$U = \sqrt{N} \, \frac{\bar{X} - \mu_0}{\hat{\sigma}'}$$

if $\mu = \mu_0$ $U$ is $T$ r.v. with $N-1$ degree of freedom

Test

$$H_0 : \mu \leq \mu_0$$

$$H_a : \mu > \mu_0$$

Reject $H_0$ if $U \geq c$ !

If we want a test of size $\alpha_0$

$$c = T_{n-1}^{-1}(1 - \alpha_0)$$

$$U \geq T_{n-1}^{-1}(1-\alpha_0)$$

$$\sqrt{n}\,\frac{\bar{X} - \mu_0}{\hat{\sigma}'} \geq T_{n-1}^{-1}(1-\alpha_0)$$

$$\bar{X} \geq \mu_0 + T_{n-1}^{-1}(1-\alpha_0)\,\frac{\hat{\sigma}'}{\sqrt{n}}$$

$$\pi(\mu, \sigma^2 \mid \delta)$$

where $\delta$ rejects $H_0$ if

$U \geq T_{n-1}^{-1}(1-\alpha_0)$.

$$\pi(\mu_0, \sigma^2 \mid \delta) = \alpha_0$$

$$\pi(\mu, \sigma^2 \mid \delta) < \alpha_0 \quad \text{if } \mu < \mu_0$$

$$\pi(\mu, \sigma^2 \mid \delta) > \alpha_0 \quad \text{if } \mu > \mu_0$$

$$\pi(\mu, \sigma^2 \mid \delta) =$$

$$\mathbb{P}(U \geq c \mid \mu, \sigma^2)$$

$$U = \sqrt{N}\,\frac{\bar{X} - \mu_0}{\hat{\sigma}'} = \underbrace{\sqrt{N}\,\frac{\bar{X} - \mu}{\hat{\sigma}'}}_{U^*} + \underbrace{\sqrt{N}\,\frac{\mu - \mu_0}{\hat{\sigma}'}}_{W}$$

$$\mathbb{P}(U \geq c \mid \mu, \sigma^2) =$$

$$\mathbb{P}(U^* \geq c - W \mid \mu, \sigma^2)$$

increasing in $\mu$.

$\pi(\mu, \sigma^2 \mid \delta) \to 1 \quad \mu \to \infty.$

p-value: if $v$ is the value of the $U$ statistics than

$$p\text{-value} = 1 - T_{n-1}(u)$$

Non-central T distribution

$$W \sim N(\psi, 1)$$

$$Y \sim \chi^2_n$$

$$X = \frac{W}{\left(\frac{Y}{n}\right)^{1/2}}$$

is called a non-central T with n d.o.f. and non-centrality $\psi$

In the T-Test the distribution of $U$ .

a non central t with non-centrality $= \sqrt{N}(\mu - \mu_0)/\sigma$

and $N-1$ d.o.f.

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$\sigma$ is Known

$$H_0 : \mu_0 \leq \mu$$

$$H_a : \mu_0 > \mu$$

$$Z = \sqrt{N}\left(\frac{\bar{X} - \mu_0}{\sigma}\right)$$

$$Z \geq \Phi^{-1}(1 - \alpha_0) = c$$

$$\pi(\mu \mid \delta) = \mathbb{P}(Z \geq c \mid \mu) =$$

$$\mathbb{P}\left(\sqrt{N}\left(\frac{\bar{X} - \mu}{\sigma}\right) + \sqrt{N}\left(\frac{\mu - \mu_0}{\sigma}\right) \geq c \mid \mu\right)$$

$$= 1 - \Phi\left(c - \sqrt{N}\left(\frac{\mu - \mu_0}{\sigma}\right)\right)$$

$\mu^* > \mu_0$ a discrepancy from

$\mu_0$ that starts beeing significant.

You can chose $N$ such that

$$\pi(\mu | \delta) > 1 - \beta_0$$

for every $\mu > \mu^*$

Since $\pi(\mu | \delta)$ is increasing it is enough to find $N$ such that

$$\pi(\mu^* | \delta) > 1 - \beta_0$$

$$1 - \Phi\left(c - \sqrt{N}\left(\frac{\mu^* - \mu_0}{\sigma}\right)\right) \geq 1 - \beta_0$$

$$\phi\left(c - \sqrt{N}\left(\frac{\mu^* - \mu_0}{\sigma}\right)\right) \leq \beta_0$$

$$c - \sqrt{N}\left(\frac{\mu_0 - \mu}{\sigma}\right) \leq \phi^{-1}(\beta_0)$$

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